

Café au Chromatic Homotopy

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Chapter 1

Preliminary

1.1 Spectra

Definition 1.1. [Swi02] A **spectrum** $E = \{(E_n, e_n, \varphi_n)\}_{n \geq 0}$ contains the following information:

- For every $n \geq 0$, a pointed CW complex (E_n, e_n)
- For every $n \geq 0$, a pointed cellular map $\varphi_n : \Sigma E_n \rightarrow E_{n+1}$

A **morphism of spectra** $f = \{f_n\}_{n \geq 0} : E = \{(E_n, e_n, \varphi_n)\}_{n \geq 0} \rightarrow F = \{(F_n, f_n, \psi_n)\}_{n \geq 0}$ contains pointed map $f_n : (E_n, e_n) \rightarrow (F_n, f_n)$ such that

$$\begin{array}{ccc} \Sigma E_n & \xrightarrow{f_n} & \Sigma F_n \\ \downarrow \Sigma \varphi_n & & \downarrow \psi_n \\ E_{n+1} & \xrightarrow{f_{n+1}} & F_{n+1} \end{array}$$

Here Ω is the loop space functor. Note the definition is equivalent to having a sequence of pointed map $\varphi'_n : \Sigma E_n \rightarrow E_{n+1}$ due to the adjunction $\Sigma \vdash \Omega : \mathcal{Spc}_* \rightarrow \mathcal{Spc}_*$, where Σ is the suspension functor, and $\mathcal{Spc}_*$ is the homotopy category of pointed spaces (we let $\mathcal{Spc}$ denote the unpointed case)

Definition 1.2. [Swi02] Given a spectrum $E := \{(E_n, \varphi_n)\}$ and a pointed CW complex X , we define the **smash product** $E \wedge X$ as $\{(E_n \wedge X, \varphi_n \wedge id_X)\}_{n \geq 0}$, where $\varphi_n \wedge id_X : E_n \wedge X \rightarrow \Omega(E_{n+1} \wedge X)$ is the adjoint of $\varphi'_n \wedge id_X : \Sigma(E_n \wedge X) = \Sigma E_n \wedge X \rightarrow E_{n+1} \wedge X$.

Definition 1.3. [Swi02] Given two spectra morphisms $f, g : E \rightarrow F$, a **homotopy** between f, g is a map $H : E \wedge I_+ \rightarrow F$, such that $H|_0 = f$ and

$H|_1 = g$, and we call f **homotopic to g** . A **homotopy equivalence** is a pair of map $f : E \rightarrow F : g$, such that $f \circ g$ is homotopic to id_F and $g \circ f$ is homotopic to id_E .

We denote $[E, F]$ the homotopy class of morphisms of spectra from E to F .

Definition 1.4. [Rud98] A **Ω -spectrum** is a spectrum $E = \{(E_n, \varphi_n)\}_{n \geq 0}$ such that the adjunction map $\varphi'_n : E_n \rightarrow \Omega E_{n+1}$ is a weak homotopy equivalence.

Notice, for a Ω -spectrum E , the stable homotopy group $\pi_k E$ is just the ordinary homotopy group of $\pi_k \Omega^\infty E$, as $\pi_{n+k} E_n \rightarrow \pi_{n+1+k} E_{n+1}$ are isomorphisms for all n .

Definition 1.5. [Swi02] A morphism of spectra $f : E \rightarrow F$ is a **weak homotopy equivalence** if f induces isomorphism $\pi_k f : \pi_k E \rightarrow \pi_k F$ for all $k \in \mathbb{Z}$ (not just nonnegative ones).

Theorem 1.6. [Rud98] *Every spectrum is weakly equivalent to a Ω -spectrum.*

Proof. Define functor $Q : \mathcal{Spc}_* \rightarrow \mathcal{Spc}_*$ as

$$Q(X) = \text{colim}_n \Omega^n \Sigma^n X$$

and for spectrum E , define $Q(E) = \{(Q(E_n), Q(\varphi_n))\}_{n \geq 0}$.

There is natural transformation $id_{\mathcal{Spc}_*} \implies Q$ from the simply inclusion, which determines a map $E \rightarrow Q(E)$. Notice $\pi_{n+k} E_n \rightarrow \pi_{n+k} Q(E_n)$ is an isomorphism for large enough k . $\square$

If one is familiar with the language of model categories, we can develop a pretty nice model category of spectra with fibrant-cofibrant objects being the Ω -spectra. Of course, everything still works even if we refrain from going into the technical details. Let us denote $\mathcal{S}$ the homotopy category of spectra, so the objects are Ω -spectra and morphisms are homotopy class.

The category $\mathcal{S}$ is pointed, so it has a zero object 0.

Definition 1.7. Define functor $\Sigma^\infty : \mathcal{Spc}_* \rightarrow \mathcal{S}$ such that $\Sigma^\infty X = \{(X_n, id)\}$, where $X_0 = X$, $X_n = \Sigma X_{n-1}$, and $id : \Sigma X_n \rightarrow X_{n+1}$ is the identity. We call this the **suspension spectrum functor** and we refer to $\Sigma^\infty X$ as the **suspension spectrum of X** .

Define functor $\Omega^\infty : \mathcal{S} \rightarrow \mathcal{Spc}_*$ such that $\Omega^\infty E = E_0$. We call this the **infinite loop space functor** and refer to $\Omega^\infty E$ as the **infinite loop space of E** .

The name suspension spectrum functor makes sense, and the infinite loop space is because $\Omega^\infty \Sigma^\infty X$ is precisely $Q(X)$ in the homotopy category.

Given a spectrum $E = \{(E_n, \varphi_n)\}_{n \geq 0}$, for every $k \in \mathbb{Z}$, there is a sequence of group homomorphisms

$$\dots \rightarrow \pi_{n+k} E_n \rightarrow \pi_{n+k} \Omega \Sigma E_n \cong \pi_{n+1+k} \Sigma E_n \xrightarrow{\pi_{n+1+k} \varphi_n} \pi_{n+1+k} E_{n+1} \rightarrow \dots$$

Where if $k < 0$, π_k is understand to be zero.

Definition 1.8. Given a spectrum E , the **stable homotopy group** $\pi_k E$ is defined as

$$\pi_k E := \operatorname{colim}_n \pi_{n+k} E_n$$

The motivation for these definitions comes from a theorem.

Theorem 1.9. (*Freudenthal's Suspension Theorem*) *Let X be a pointed n -connected CW complex, then for any k , the map*

$$\pi_k X \rightarrow \pi_k \Omega \Sigma X = \pi_{k+1} \Sigma X$$

induced from the adjunction unit is an isomorphism if $k \leq 2n$ and epimorphism if $k = 2n + 1$.

Suppose we have any pointed CW complex X , $\Sigma^n X$ will be at least $n - 1$ -connected. Therefore, the map

$$\pi_{n+k} \Sigma^n X \rightarrow \pi_{(n+1)+k} \Sigma^{n+1} X$$

is an isomorphism if $n + k \leq 2(n - 1)$, or equivalently, $k \leq n - 2$. Moreover,

$$\pi_{(n+1)+k} \Sigma^{n+1} X \rightarrow \pi_{(n+2)+k} \Sigma^{n+2} X$$

is an isomorphism if $k \leq n - 1$.

In other words, as we go down the sequence, the homotopy groups will *stabilize*. Therefore, the stable homotopy group $\pi_k \Sigma^\infty X$ is just $\pi_k X_m$ for large enough m .

Definition 1.10. We define functor $\Omega : \mathcal{S} \rightarrow \mathcal{S}$ such that $\Omega E = \{(E_{n+1}, \varphi_{n+1})\}_{n+1 \geq 0}$, so we shift everything down. There is functor $\Sigma : \mathcal{S} \rightarrow \mathcal{S}$ such that $\Sigma E = \{(E_{n-1}, \varphi_{n-1})\}_{n-1 \geq 0}$, where we define $E_{-1} = \Omega E_0$.

Notice, there is $\Omega \Sigma \simeq \Sigma \Omega \simeq id$.

We denote $S := \Sigma^\infty S^0$. Since we have $\Sigma \vdash \Omega : \mathcal{S}pc_* \rightarrow \mathcal{S}pc_*$, for any spectra E , there is $\pi_k E \cong [\Sigma^k S, E]$.

Theorem 1.11. *There exists adjunction $\Sigma^\infty \vdash \Omega^\infty : \mathcal{Spc}_* \rightarrow \mathcal{S}$, i.e. $[\Sigma^\infty X, E] \cong [X, \Omega^\infty E]_*$*

Proof. Given $f : \Sigma^\infty X \rightarrow E$, there is

$$\begin{array}{ccc} \Sigma \Sigma^n X & \xrightarrow{\Sigma f_n} & \Sigma E_n \\ \Downarrow & & \downarrow \varphi_n \\ \Sigma^{n+1} X & \xrightarrow{f_{n+1}} & E_{n+1} \end{array}$$

Notice, that f_0 determines f_n uniquely. $\square$

The details of definitions below require the knowledge of the model structure on the category of spectra, which is not necessary for understanding the note.

Definition 1.12. Let $F : I \rightarrow \mathcal{S}$ be a homotopy coherent diagram of spectra, we denote $\text{holim}_I F$ the **homotopy limit** of F in the model category of spectra, and $\text{hocolim}_I F$ the **homotopy colimit** in the model category of spectra.

We specifically look at the homotopy cofiber sequence. For every $f : E \rightarrow F$, we define $Cf := \{(F_n \cup_{f_n} CE_n, \varphi_n)\}$, where φ_n is induced from the structure map $\Sigma F_n \rightarrow F_{n+1}$ and $\Sigma(CE_n) = C(\Sigma E_n) \rightarrow CE_{n+1}$.

Proposition 1.13. *[Swi02] For every morphism $f : E \rightarrow F$, there is a homotopy cofiber sequence*

$$E \xrightarrow{f} F \rightarrow Cf \rightarrow \Sigma E$$

Proof. Homotopy cofiber in a simplicial model category (for instance $\mathcal{S}$) is just ordinary pushout with the point in the diagram replacement with the cone, so Cf determines the homotopy cofiber of f .

We have homotopy pasting law, so we just need to show the homotopy pushout of E is ΣE (which we guess is true from the notation chosen). We have that the homotopy pushout will be E' where $E'_0 = *$ and $E'_n = \Sigma E_{n-1}$. There is map $f : E' \rightarrow \Sigma E$ induced from the structure map $\Sigma E_{n-1} \rightarrow E_n$, which we know induces isomorphism on the stable homotopy groups. $\square$

We have a similar result regarding homotopy fiber. Given map $f : E \rightarrow F$, let Pf be the homotopy fiber of f . There is then homotopy fiber sequence

$$\Omega F \rightarrow Pf \rightarrow E \xrightarrow{f} F$$

Proposition 1.14. *A sequence $E \rightarrow F \rightarrow G$ is a homotopy cofiber sequence if and only if it is a homotopy fiber sequence.*

In other words, we can extend every cofiber (equivalently fiber) sequence $E \rightarrow F \rightarrow G$ into a long exact sequence

$$\dots \rightarrow \Omega F \rightarrow \Omega G \rightarrow E \rightarrow F \rightarrow G \rightarrow \Sigma E \rightarrow \Sigma F \rightarrow \dots$$

where any two adjacent map determines a cofiber/fiber sequence.

Since every spectrum is a looping of its suspension, so the homotopy class of spectra morphisms are abelian groups (in fact, the suspension and loop space functor, with the additive property of the homotopy category of spectra determines a triangulated category)

Proposition 1.15. *Suppose $E \rightarrow F \rightarrow G$ is a cofiber sequence, then for any spectrum X*

$$\begin{aligned} [X, E] &\rightarrow [X, F] \rightarrow [X, G] \\ [G, X] &\rightarrow [F, X] \rightarrow [E, X] \end{aligned}$$

are exact sequences of abelian groups.

Theorem 1.16. *[Rud98] There exists a symmetric monoidal structure $\wedge$ on $\mathcal{S}$ such that*

- *There are natural equivalences determining the symmetric monoidal structure*

$$\begin{aligned} \alpha_{E,F,G} : (E \wedge F) \wedge G &\rightarrow E \wedge (F \wedge G) \\ \tau_{E,F} : E \wedge F &\rightarrow F \wedge E \\ l_E : S \wedge E &\rightarrow E \\ r_E : E \wedge S &\rightarrow E \end{aligned}$$

- *There are natural equivalence $e_{X,E} : E \wedge X \rightarrow E \wedge \Sigma^\infty X$.*
- *For every diagram $F : I \rightarrow \mathcal{S}$, the canonical morphism $(\text{hocolim}_I F) \wedge E \rightarrow \text{hocolim}_I (F \wedge E)$ is an equivalence*

Definition 1.17. *[Swi02] A ring spectrum is a tuple (E, ψ, e) with $\psi : E \wedge E \rightarrow E$ and $e : S \rightarrow E$, such that*

$$\begin{array}{ccc} (E \wedge E) \wedge E & \xrightarrow{\alpha_{E,E,E}} & E \wedge (E \wedge E) \xrightarrow{id_E \wedge \psi} E \wedge E \\ \downarrow \psi \wedge id_E & & \downarrow \psi \\ E \wedge E & \xrightarrow{\psi} & E \end{array}$$

and

$$\begin{array}{ccccc}
 S \wedge E & \xrightarrow{e \wedge id_E} & E \wedge E & \xleftarrow{id_E \wedge e} & E \wedge S \\
 \downarrow l_E & & \downarrow \psi & & \downarrow r_E \\
 E & \xrightleftharpoons{\quad} & E & \xleftarrow{\quad} & E
 \end{array}$$

commutes (up to homotopy). Moreover, if

$$\begin{array}{ccc}
 E \wedge E & \xrightarrow{\tau_{E,E}} & E \wedge E \\
 & \searrow \psi & \downarrow \psi \\
 & & E
 \end{array}$$

commutes (up to homotopy), then we call it a **commutative ring spectrum**.

Definition 1.18. Given a ring spectra (E, ψ, e) , a left (resp. right) E -module is a tuple (M, ϕ) , where $\phi : E \wedge M \rightarrow M$ (resp. $\phi : M \wedge E \rightarrow M$) such that

$$\begin{array}{ccc}
 (E \wedge E) \wedge M & \xrightarrow{\alpha_{E,E,M}} & E \wedge (E \wedge M) \xrightarrow{id_E \wedge \phi} E \wedge M \\
 \downarrow \psi \wedge id_M & & \downarrow \phi \\
 E \wedge M & \xrightarrow{\quad \phi \quad} & M
 \end{array}
 \qquad
 \begin{array}{ccc}
 (M \wedge E) \wedge E & \xrightarrow{\alpha_{M,E,E}} & M \wedge (E \wedge E) \xrightarrow{id \wedge \psi} M \wedge E \\
 \downarrow \phi \wedge id & & \downarrow \phi \\
 M \wedge E & \xrightarrow{\quad \phi \quad} & M
 \end{array}$$

the left diagram commute up to homotopy (resp. right diagram commute up to homotopy).

1.2 Cohomology

Definition 1.19. [Rud98] A **reduced generalized cohomology** is a contravariant functor $\mathcal{H}^* : \mathcal{Spc}_*^{op} \rightarrow \text{grAb}$ from the homotopy category of spaces to the category of $\mathbb{Z}$ -graded abelian groups, such that

- (Additive) For every set of pointed spaces X_α for $\alpha \in A$, the canonical homomorphism $\mathcal{H}^*(\bigvee_{\alpha \in A} X_\alpha) \rightarrow \prod_{\alpha \in A} \mathcal{H}^*(X_\alpha)$ is an isomorphism.
- (Mayer-Vietoris) For every homotopy cofiber sequence in pointed space $X \rightarrow Y \rightarrow Z$, there is long exact sequence

$$\dots \rightarrow \mathcal{H}^{n-1}(X) \xrightarrow{\delta_{n-1}} \mathcal{H}^n(Z) \rightarrow \mathcal{H}^n(Y) \rightarrow \mathcal{H}^n(X) \xrightarrow{\delta_n} \mathcal{H}^{n+1}(Z) \rightarrow \dots$$

A **reduced generalized homology** is a covariant functor $\mathcal{H}_* : \mathcal{Spc}_* \rightarrow \mathbf{grAb}$ from the homotopy category of spaces to the category of $\mathbb{Z}$ -graded abelian groups, such that

- (Additive) For every set of pointed spaces X_α for $\alpha \in A$, the canonical homomorphism $\coprod_{\alpha \in A} \mathcal{H}_*(X_\alpha) \rightarrow \mathcal{H}_*(\bigvee_{\alpha \in A} X_\alpha)$ is an isomorphism.
- (Mayer-Vietoris) For every homotopy cofiber sequence in pointed space $X \rightarrow Y \rightarrow Z$, there is long exact sequence

$$\dots \rightarrow \mathcal{H}^{n+1}(Z) \xrightarrow{\delta_n} \mathcal{H}_n(X) \rightarrow \mathcal{H}_n(Y) \rightarrow \mathcal{H}_n(Z) \xrightarrow{\delta_{n-1}} \mathcal{H}_{n-1}(X) \rightarrow \dots$$

A morphism of cohomology (resp. homology) are just natural transformations.

Remark 1.20. Given a spectrum E , the functor $E^n(X) := [\Sigma^{-n}\Sigma^\infty X, E]$ (where $\Sigma^{-n} = \Omega^n$ for $n > 0$), is a cohomology.

The additive property follows from

$$[\Sigma^{-n}\Sigma^\infty(\bigvee_{\alpha \in A} X_\alpha), E] \cong [\bigvee_{\alpha \in A} \Sigma^{-n}\Sigma^\infty(X_\alpha), E] \cong \prod_{\alpha \in A} [\Sigma^{-n}\Sigma^\infty(X_\alpha), E]$$

The Mayer-Vietoris property follows from that a homotopy cofiber sequence is taken to homotopy cofiber sequence

$$\dots \rightarrow \Sigma^{-(n+1)}\Sigma^i nfty Z \rightarrow \Sigma^{-n}\Sigma^i nfty X \rightarrow \Sigma^{-n}\Sigma^i nfty Y \rightarrow \Sigma^{-n}\Sigma^i nfty Z \rightarrow \Sigma^{-(n-1)}\Sigma^i nfty X \rightarrow \dots$$

Then there is long exact sequence

$$\dots \rightarrow E^{n-1}(X) = [\Sigma^{-(n-1)}\Sigma^i nfty X, E] \rightarrow E^n(Z) = [\Sigma^{-n}\Sigma^i nfty Z, E] \rightarrow E^n(Y) \rightarrow E^n(Z) \rightarrow E^{n-1}(X)$$

Equivalently, the functor $E_n(X) := [\Sigma^n, \Sigma^\infty X \wedge E]$ is a generalized homology.

Theorem 1.21. (*Brown's Representability Theorem*) [[Swi02](#)]

The category of reduced generalized cohomologies is equivalent to the homotopy category of spectra from the construction above.

Remark 1.22. Warning: the generalized homology is not equivalent to spectra. There exists morphism of homology theories that does not determine a spectra map.

1.3 Examples

1.4 Operations and cooperations

1.5 Localization

1.6 Adams–Novikov spectral sequence

1.7 Computational tools

Chapter 2

Chromatic Homotopy

2.1 Complex orientations

Definition 2.1. A **complex orientable** spectrum is a multiplicative spectrum E such that $\tilde{E}^2(\mathbb{C}P^\infty) \rightarrow \tilde{E}^2(\mathbb{C}P^1) \cong E^0$ is surjective.

Since E is multiplicative, E^0 has a generator 1_E , which correspond to a generator $t \in E^2(\mathbb{C}P^1)$. A **complex orientation** is a lift of t in $E^2(\mathbb{C}P^\infty)$. A **complex oriented cohomology** is a tuple (E, t_E) where E is a complex orientable cohomology with a complex orientation

We have a map $E^*[t]/(t^{n+1}) \rightarrow E^*(\mathbb{C}P^n)$, from the fact that there is a cover of $n + 1$ contractible open set of $\mathbb{C}P^n$. As it turns out

Lemma 2.2. [Hop99] *The map is an isomorphism.*

Proof. For $E = H\mathbb{Z}$, this is a classic result. We generalize it using the Atiyah-Hirzebruch SS, which is multiplicative, with one generator t that correspond to the element $t \in E^2(\mathbb{C}P^n)$. We have that $\square$

We can take one step further, from Milnor's sequence

$$0 \rightarrow \lim^1 E^{*-1}(\mathbb{C}P^n) \rightarrow E^*(\mathbb{C}P^\infty) \rightarrow \lim E^*(\mathbb{C}P^n) \rightarrow 0$$

and the fact $E^*(\mathbb{C}P^{n+1}) \rightarrow E^*(\mathbb{C}P^n)$ is surjective, we have

$$E^*(\mathbb{C}P^\infty) \cong E^*[[t]]$$

The name complex orientation has a geometric meaning. We know complex line bundle is classified by the space $\mathbb{C}P^\infty \simeq BU(1)$. Given a complex

oriented cohomology, t is then a characteristic class of universal complex line bundle.

The tensor product of line bundle $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \rightarrow \mathbb{C}P^\infty$ then induces a map

$$\psi_E : E^*(\mathbb{C}P^\infty) \cong E^*[[t]] \rightarrow E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong E^*[[x, y]]$$

The generator t then correspond to a formal power series in x, y , $F_E(x, y) := \psi_E(t)$.

Definition 2.3. A formal group law over a ring R is a formal power series $F(x, y)$ such that

- $F(x, 0) = F(0, x) = x$
- $F(x, y) = F(y, x)$
- $F(x, F(y, z)) = F(F(x, y), z)$

For R graded, we require $F(x, y)$ to be homogeneous of degree 2.

We will write $x +_F y := F(x, y)$.

Lemma 2.4. [Rud98] Let E be a complex oriented cohomology, $F_E(x, y)$ is a graded formal group law.

Proof. This follows from the properties of tensor product of line bundle being unital, associative and commutative. $\square$

Example 2.5. Let $E = HR$ for some ring R , $f_E(x, y) = x + y$, as it is the only possible graded formal group law.

Let $E = K$ be the complex K -theory, we choose $t \in K^2(\mathbb{C}P^\infty)$ to be the one that correspond to $[\mathcal{O}(1)] - 1$, where $\mathcal{O}(1)$ is the universal line bundle and 1 is the trivial rank 1 bundle over $\mathbb{C}P^\infty$, under the bott map $\beta : K^* \rightarrow K^{*+2}$. We have that given line bundle

$$[\mathcal{O}(1)] \otimes [\mathcal{O}(1)] - 1 = ([\mathcal{O}(1)] - 1) \otimes ([\mathcal{O}(1)] - 1) + ([\mathcal{O}(1)] - 1) \otimes 1 + 1 \otimes ([\mathcal{O}(1)] - 1)$$

So $F_K(x, y) = x + y + \beta^{-1}xy$

2.2 Computing MU

2.3 Formal group laws

2.4 Universal orientations

Chapter 3

Algebraic Geometry

3.1 Schemes

3.2 Morphisms

3.3 Topology

3.4 Quasicoherent modules

3.5 Formal schemes

3.6 Stacks

Chapter 4

Formal Groups

4.1 Coordinates

4.2 Formal group laws

4.3 Height

4.4 Invariant differentials

4.5 Moduli of formal groups

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